STABILITY AND CONVERGENCE OF THE MODERN TAYLOR SERIES METHOD

Jiří Kunovský, Václav Šátek, Pavla Sehnalová

Brno University of Technology, Faculty of Information Technology
Božetěchova 2, CZ-612 66 Brno, Czech Republic

kunovsky@fit.vutbr.cz(Jiří Kunovský)

Abstract

The paper deals with extremely exact, stable and fast numerical solutions of systems of differential equations. In a natural way, it also involves solutions of problems that can be transformed to solving a system of differential equations.
The project is based on an original mathematical method which uses the Taylor series method for solving differential equations.
The Taylor Series Method is based on a recurrent calculation of the Taylor series terms for each time interval. Thus the complicated calculation of higher order derivatives (much criticized in the literature) need not be performed but rather the value of each Taylor series term is numerically calculated. Another typical algorithm is the convolution operation. Stability and convergence of the numerical integration methods when the Dahlquist problem is solved, Taylorian initial problems with automatic transformation, stability and convergence of a system of linear algebraic equations and stability and convergence when algebraic and transcendental equations are solved will be discussed in the paper.

Keywords: Stability, Convergence, Modern Taylor Series Method, Differential equations, Continuous system modelling

Presenting Author's Biography

Jiří Kunovský graduated at Brno University of Technology, in 1967. During most of his time at BUT he has taught and directed research in Computer Science, specially in simulations of “Security-Oriented Research in Information Technology”. He has created the simulation language TKSL. He has presented the work on the Modern Taylor Series Method and the simulation language TKSL at many occasions at home and abroad.

1 Introduction - Taylorian Initial Problem

An important part of the newly developed Taylor Series Method (Modern Taylor Series Method) is an automatic integration order setting, i.e. using as many Taylor series terms as the defined accuracy requires. Thus it is usual that the computation uses different numbers of Taylor series terms for different steps of constant length. On the other hand, for a pre-set integration order, the integration step length may be selected. This fact positively affects the stability and speed of the computation.

An automatic transformation of the original problem (when stability and convergence are computed) is a necessary part of the Modern Taylor Series Method. The original system of differential equations is automatically transformed to a polynomial form, i.e. to a form suitable for easily calculating the Taylor series forms using recurrent formulae.
As an example, a transformation into polynomial form the following initial problem

$$x_1' = \frac{1}{\sin x_1}, \quad x_1(0) = \frac{\pi}{2} \tag{1}$$

is presented. Putting

$$\frac{1}{\sin x_1} = x_2$$

we can construct

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= \frac{\pi}{2} \\ x_2' &= -x_2^3 \cos(x_1) & x_2(0) &= 1 \end{aligned} \tag{2}$$

Putting $x_3 = \cos(x_1),\ x_4 = \sin(x_1)$ we obtain a Taylorian initial problem

$$\begin{aligned} x_1' &= x_2 & x_1(0) &= \frac{\pi}{2} \\ x_2' &= -x_2^3 \cdot x_3 & x_2(0) &= 1 \\ x_3' &= -x_2 \cdot x_4 & x_3(0) &= 0 \\ x_4' &= x_2 \cdot x_3 & x_4(0) &= 1 \end{aligned} \tag{3}$$

We can see that all the expressions on the right-hand sides of (3) are polynomials.

The idea of the automatic transformation requires the software capable of automatic performing the decomposition of the right-hand sides of ordinary differential equations. This new approach has been implemented in a simulation language TKSL (an implementation of the Modern Taylor Series Method on a personal computer). In fact, the well-known rules of differential and integral calculus have been used.

2 Stability and Convergence

Both attributes, the stability and convergence have been analyzed for well-known Dahlquist problem [1]

$$y' = \lambda y, \quad y_0 = 1, \quad \lambda < 0 \tag{4}$$

with well-known exact solution

$$y = y_0 e^{\lambda t} \tag{5}$$

Numerical solution using explicit Taylor series is in the form

$$\begin{aligned} y_{n+1} &= y_n + h y_n' + \frac{h^2}{2} y_n'' + \cdots + \frac{h^k}{k!} y_n^{(k)} \\ y_{n+1} &= y_n + h\lambda y_n + \frac{h^2}{2}\lambda^2 y_n + \cdots + \frac{h^k}{k!}\lambda^k y_n \\ y_{n+1} &= \sum_{i=0}^{k} \frac{(\lambda h)^i}{i!} y_n \end{aligned} \tag{6}$$

where h is an integration step.

Similarly implicit Taylor series is in the form

$$\begin{aligned} y_{n+1} &= y_n + h y_{n+1}' + \frac{h^2}{2} y_{n+1}'' + \cdots + \frac{h^k}{k!} y_{n+1}^{(k)} \\ y_{n+1} &= y_n + h\lambda y_{n+1} + \frac{h^2}{2}\lambda^2 y_{n+1} + \cdots + \frac{h^k}{k!}\lambda^k y_{n+1} \\ y_{n+1} &= (\sum_{i=0}^{k} \frac{(-\lambda h)^i}{i!} y_n)^{-1} \end{aligned} \tag{7}$$

Let $z = \lambda h$, then the stability function is $R(z) = \frac{y_{n+1}}{y_n}$. The stability domain is defined as $R(z) < 1$. Stability domain for some well-known numerical methods are in Tab. 1.

Tab. 1 Stability functions

method	$R(z)$
explicit Euler method	$1 + z$
implicit Euler method	$\frac{1}{1-z}$
Trapezoidal method	$\frac{1+z/2}{1-z/2}$
explicit Taylor method	$1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!}$
implicit Taylor method	$\frac{1}{1-z-\frac{z^2}{2}-\cdots-\frac{z^k}{k!}}$

Eq. (4) becomes stiff if $\lambda << 0$. In all computations, the constant $\lambda = -100$. The general approach to the solution of stiff differential equations is to use implicit numerical methods. Numerical methods and their stability and convergence will be a part of this section. Typical results of our analysis are in Fig. 1 - 9. Functions of time of numerical solutions of the Dahlquist problem for exact solution, implicit Taylor method ($ORD = 9$), implicit Euler method ($ORD = 1$) and Trapezoidal method are presented in Fig. 1. An abbreviation ORD is used for the method order (for example $ORD = 9$ means that nine terms of the Taylor series are used for computation). As expected, there are some oscillation when Trapezoidal method is used. Implicit Taylor of $ORD = 9$ has nearly the same quality of computation as the exact solution of the Dahlquist problem (5).

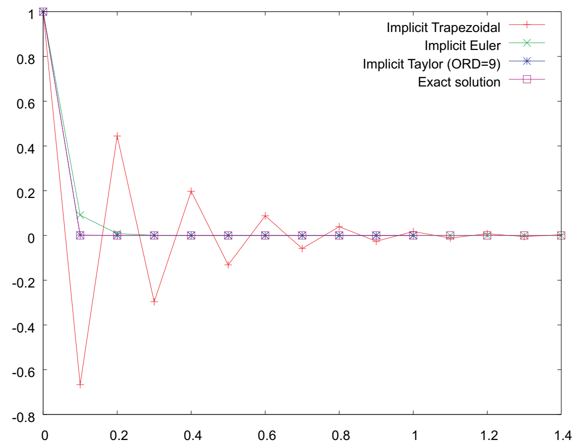

Fig. 1 Numerical solution of Dahlquist problem ($\lambda = -100, h = 0.1$)

Well-known stability domain of the implicit Euler method is presented in Fig. 2.

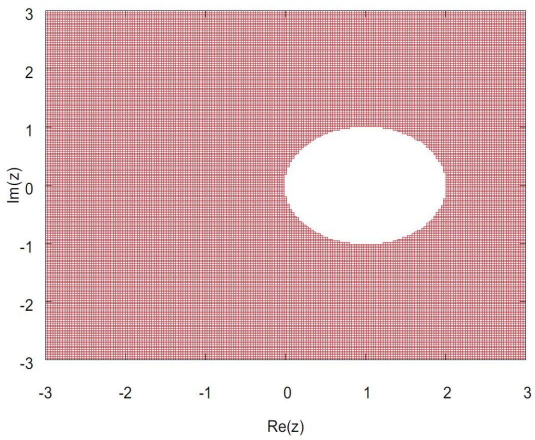

Fig. 2 Stability domain for implicit Euler

Similarly, expected stability domains of the Trapezoidal method and the implicit Taylor method ($ORD = 9$) are in Fig. 3 and Fig. 4.

On the contrary, the well-known results of the explicit integration methods are presented in Fig. 5 (explicit Euler), in Fig. 6 (explicit Taylor of $ORD = 2$) and in Fig. 7 (explicit Taylor of $ORD = 3$).

The comparison of stability domains of the explicit Taylor method of $ORD = 63$ and implicit Taylor method of $ORD = 63$ can be seen in Fig. 8 and Fig. 9. Details will be discussed during the presentation.

3 Systems of Linear Algebraic Equations

In the previous section the stability domains of chosen methods have been presented. In this section the stability and the idea of solving the system of linear algebraic equations will be discussed. Again the stability and convergence of the system are closely connected.

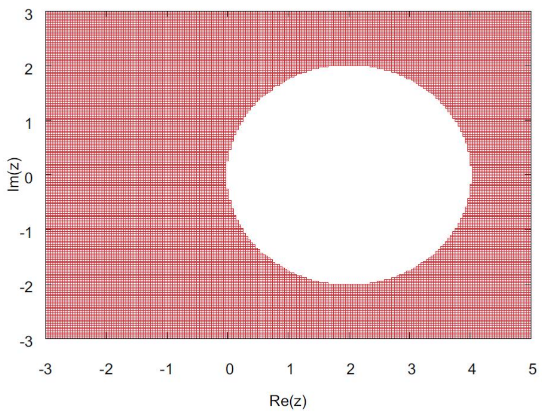

Fig. 3 Stability domain for Trapezoidal method

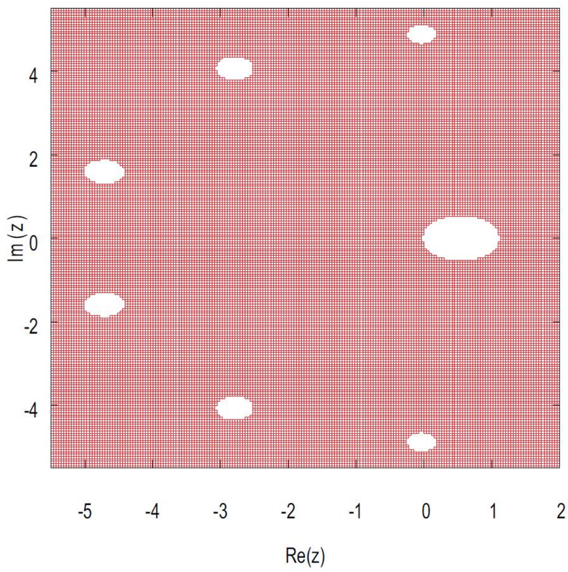

Fig. 4 Stability domain for implicit Taylor of $ORD = 9$

Systems of linear algebraic equations (SLAE) can be also solved by the Modern Taylor Series Method. In this case the system of linear algebraic equations

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0} \tag{8}$$

must be transformed to a system of differential equations (SDE)

$$\mathbf{Ax} - \mathbf{b} = -\mathbf{x}'. \tag{9}$$

Supposing that the real parts of roots of the characteristic equation

$$\mid \mathbf{A} - \lambda\mathbf{I} \mid = 0, \tag{10}$$

where $\mathbf{I}$ is the unit matrix, are negative, the derivatives on the right-hand side of the system (9) will be equal to zero in a stable state and the solution of the SLAE will be identical with the solution of the system of differential equations (SDE). It has been shown that the solution of the regular matrix A of the system of linear algebraic equations is stable and it converges to the expected (exact) solution.

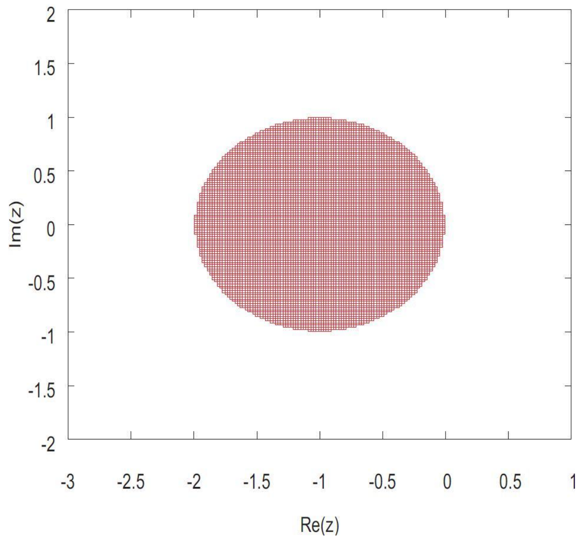

Fig. 5 Stability domain for explicit Euler

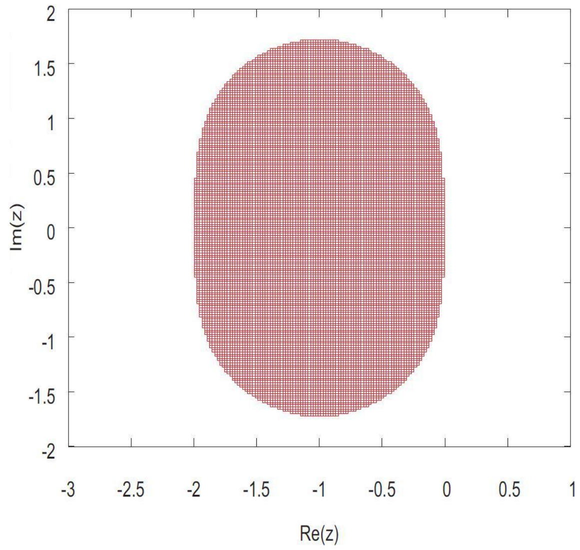

Fig. 6 Stability domain for explicit Taylor of $ORD = 2$

Since not every matrix $\mathbf{A}$ satisfies the condition (10), the system of differential equations (8) has to be transformed to a stable system. One of the ways to do this is to multiply the whole system of algebraic equations by transposed matrix $\mathbf{A}^T$ from left, so that the actual system to be solved is

$$\mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{A}^T\mathbf{b} = -\mathbf{x}'. \tag{11}$$

If the matrix $\mathbf{A}$ is non-singular, which is a general condition for a SLAE to have a solution, the resulting matrix $\mathbf{A}^T\mathbf{A}$ is positively definite.

The matrix $\mathbf{A}$ is real, thus $\mathbf{A}^T\mathbf{A}$ is positively stable and so is the system (11). A transformation performed by multiplying $\mathbf{A}$ by the transposed matrix $\mathbf{A}^T$ has a special property - the resulting functions resemble a strong attenuation.

The source code of the TKSL solution of 10 algebraic

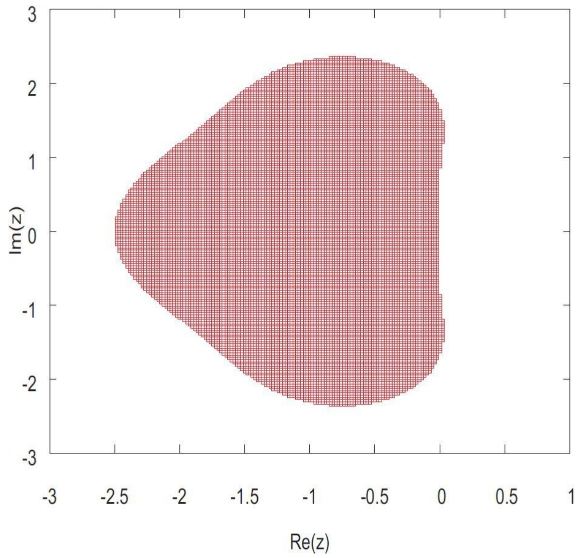

Fig. 7 Stability domain for explicit Taylor of $ORD = 3$

equations is in 3.1 Example. An easy strategy of multiplication by transposed matrix can also be seen.

3.1 Example

```
var
q1, q2, q3, q4, q5, q6, q7, q8, q9, qA,
x1, x2, x3, x4, x5, x6, x7, x8, x9, xA;

const
dt=0.1, eps=1e-20, tmax=29,

a11=1, a12= 1, a13= 1, a14= 1, a15= 1, a16= 1,
a17= 1, a18= 1, a19= 2, a1A=3,  b1=-5,
a21=1, a22=-2, a23= 1, a24= 4, a25= 1, a26=-5,
a27= 1, a28= 3, a29= 1, a2A=1,  b2= 7,
a31=2, a32= 1, a33= 3, a34= 1, a35=-1, a36= 1,
a37=-3, a38= 1, a39= 4, a3A=1,  b3=-2,
a41=1, a42=-1, a43= 1, a44=-1, a45= 1, a46=-3,
a47= 1, a48= 1, a49=-1, a4A=1,  b4= 8,
a51=1, a52= 2, a53= 3, a54=-3, a55=-2, a56=-1,
a57= 1, a58= 2, a59= 3, a5A=4,  b5=-1,
a61=5, a62= 3, a63= 1, a64= 1, a65= 3, a66= 2,
a67= 1, a68= 1, a69= 1, a6A=1,  b6= 5,
a71=3, a72= 1, a73= 1, a74= 2, a75= 3, a76= 3,
a77= 2, a78= 1, a79= 1, a7A=2,  b7=-1,
a81=4, a82= 5, a83=-2, a84=-2, a85= 2, a86= 6,
a87= 3, a88=-1, a89= 1, a8A=1,  b8= 3,
a91=1, a92= 1, a93= 7, a94=-3, a95= 2, a96= 1,
a97= 2, a98=-3, a99=-5, a9A=2,  b9= 8,
aA1=7, aA2=-2, aA3=-2, aA4=-3, aA5=-4, aA6=-3,
aA7=-1, aA8= 1, aA9= 2, aAA=1,  bA= 5;

system
q1 = a11*x1+a12*x2+a13*x3+a14*x4+a15*x5+a16*x6
     +a17*x7+a18*x8+a19*x9+a1A*xA-b1;
q2 = a21*x1+a22*x2+a23*x3+a24*x4+a25*x5+a26*x6
     +a27*x7+a28*x8+a29*x9+a2A*xA-b2;
q3 = a31*x1+a32*x2+a33*x3+a34*x4+a35*x5+a36*x6
     +a37*x7+a38*x8+a39*x9+a3A*xA-b3;
q4 = a41*x1+a42*x2+a43*x3+a44*x4+a45*x5+a46*x6
     +a47*x7+a48*x8+a49*x9+a4A*xA-b4;
q5 = a51*x1+a52*x2+a53*x3+a54*x4+a55*x5+a56*x6
     +a57*x7+a58*x8+a59*x9+a5A*xA-b5;
q6 = a61*x1+a62*x2+a63*x3+a64*x4+a65*x5+a66*x6
     +a67*x7+a68*x8+a69*x9+a6A*xA-b6;
```

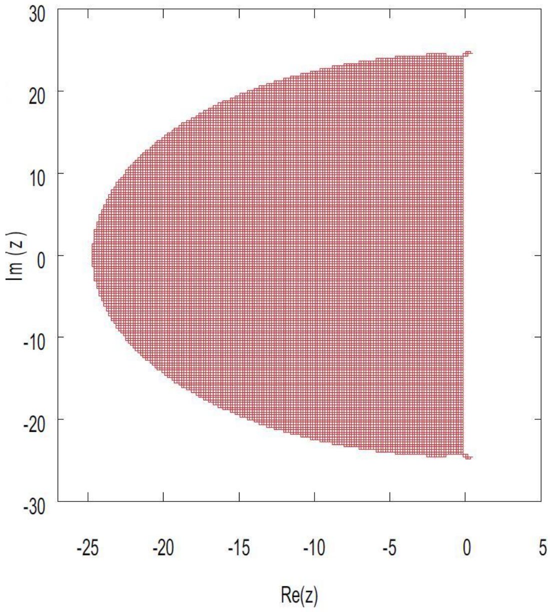

Fig. 8 Stability domain for explicit Taylor $ORD = 63$

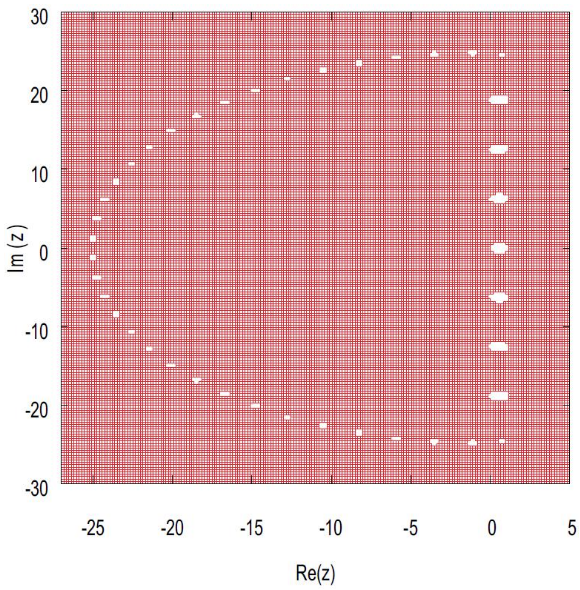

Fig. 9 Stability domain for implicit Taylor $ORD = 63$

```
q7 = a71*x1+a72*x2+a73*x3+a74*x4+a75*x5+a76*x6
      +a77*x7+a78*x8+a79*x9+a7A*xA-b7;
q8 = a81*x1+a82*x2+a83*x3+a84*x4+a85*x5+a86*x6
      +a87*x7+a88*x8+a89*x9+a8A*xA-b8;
q9 = a91*x1+a92*x2+a93*x3+a94*x4+a95*x5+a96*x6
      +a97*x7+a98*x8+a99*x9+a9A*xA-b9;
qA = aA1*x1+aA2*x2+aA3*x3+aA4*x4+aA5*x5+aA6*x6
      +aA7*x7+aA8*x8+aA9*x9+aAA*xA-bA;

x1'= -(a11*q1+a21*q2+a31*q3+a41*q4+a51*q5+
      a61*q6+a71*q7+a81*q8+a91*q9+aA1*qA) &0;
x2'= -(a12*q1+a22*q2+a32*q3+a42*q4+a52*q5+
      a62*q6+a72*q7+a82*q8+a92*q9+aA2*qA) &0;
x3'= -(a13*q1+a23*q2+a33*q3+a43*q4+a53*q5+
      a63*q6+a73*q7+a83*q8+a93*q9+aA3*qA) &0;
x4'= -(a14*q1+a24*q2+a34*q3+a44*q4+a54*q5+
      a64*q6+a74*q7+a84*q8+a94*q9+aA4*qA) &0;
x5'= -(a15*q1+a25*q2+a35*q3+a45*q4+a55*q5+
      a65*q6+a75*q7+a85*q8+a95*q9+aA5*qA) &0;
x6'= -(a16*q1+a26*q2+a36*q3+a46*q4+a56*q5+
      a66*q6+a76*q7+a86*q8+a96*q9+aA6*qA) &0;
x7'= -(a17*q1+a27*q2+a37*q3+a47*q4+a57*q5+
      a67*q6+a77*q7+a87*q8+a97*q9+aA7*qA) &0;
x8'= -(a18*q1+a28*q2+a38*q3+a48*q4+a58*q5+
      a68*q6+a78*q7+a88*q8+a98*q9+aA8*qA) &0;
x9'= -(a19*q1+a29*q2+a39*q3+a49*q4+a59*q5+
      a69*q6+a79*q7+a89*q8+a99*q9+aA9*qA) &0;
xA'= -(a1A*q1+a2A*q2+a3A*q3+a4A*q4+a5A*q5+
      a6A*q6+a7A*q7+a8A*q8+a9A*q9+aAA*qA) &0;
sysend.
```

Results of solutions of the 3.1 Example are in Fig. 10. A nice property of the method can be specified, the stable state for regular matrix does not depend on initial conditions. Thus it is very easy to confirm the same stable state result for any change of initial conditions. This might be useful for large systems of algebraic equations. For the completeness, the case of singular matrix A of the system of linear algebraic equations is also presented.

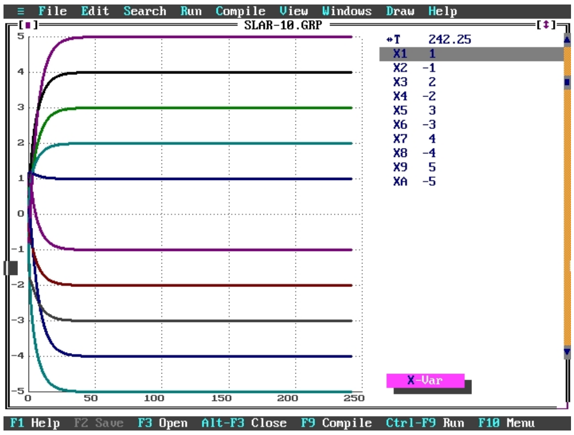

Fig. 10 Regular matrix

The result of solution of system of algebraic equations with singular matrix can be seen in Fig. 11. All variables are the same as in 3.1 Example, only new coefficients are set to $a11 = 1, aA2 = a12, \cdots, aAA = a1A, bA = b1$. Solutions for zero initial conditions are at the upper part of Fig. 11. Solutions for all initials conditions set to -10 are at the bottom part of Fig. 11. There are reasonable changes in stable states and thus singular system is detected.

4 Algebraic and Transcendental Equations

In this chapter a special method to find the real roots of an explicit set of algebraic (or transcendental) equations is described.

When applicable, the simplest method to obtain solutions $f(x) = 0$ is to draw a graph of $f(x)$ and read the roots.

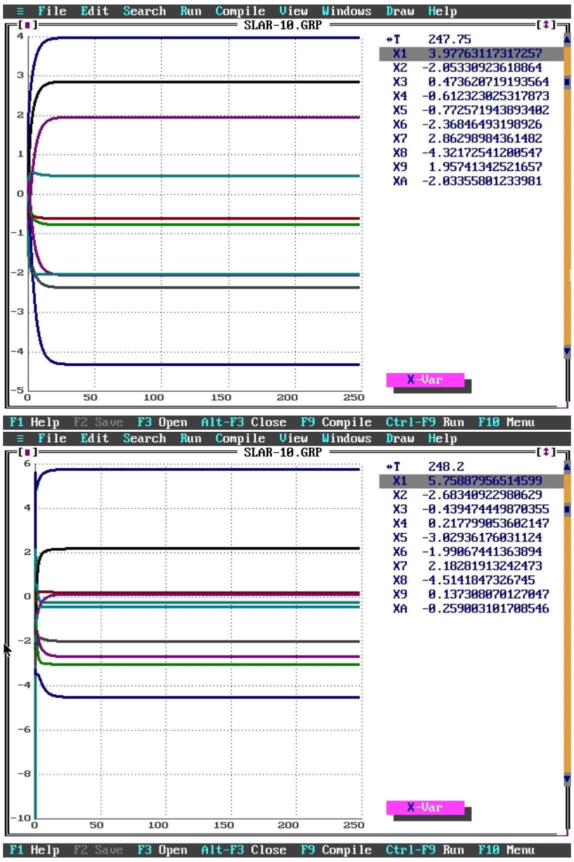

Fig. 11 Singular matrix

As an example the Chebyshev polynom

$$f(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$

is analyzed for $x \epsilon < 0, 1 >$.

The first root of the equation is obviously x_1=0.

Computation with an automatic stop at finding roots can be used with advantage. The following nonlinear equation of movement pertaining to the solution x is defined as

$$\frac{dx}{dt} = \lambda f(x), \qquad (12)$$

when λ is a suitable positive or negative number. The root is found when $\frac{dx}{dt} = 0$.

The following computation scheme is used:

- λ is set at λ = + 1. The system (12) is displaced from the initial stable state (from the first equation root x_1 = 0) to a new "non-stable" state such as x(0)=0.005. Fig. 12 plots the substitute solution x (for x(0)=0.005). The new stable state $x_2 = 0.342020143325669$ is found.
- λ is set at λ = - 1. The system (12) is displaced from the previous stable state (from the equation root x_2 = 0.342020143325669) to a new "non-stable" state such as x(0)=0.36 and x_3 = 0.642787609686539 is found.

Similarly x_4 = 0.866025403784439 and x_5 = 0.984807753012208 have been obtained.

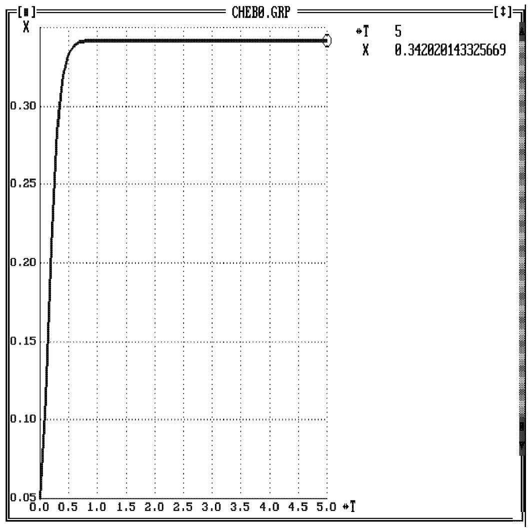

Fig. 12 An automatic stop

5 Summary

Stability and convergence of the numerical integration methods for the Dahlquist problem, Taylorian initial problems with automatic transformation, the system of linear algebraic equations and algebraic and transcendental equations have been discussed in the paper. The "Modern Taylor Series Method" also has some properties very favourable for parallel processing. Many calculation operations are independent making it possible to perform the calculations independently using separate processors of parallel computing systems.

6 Acknowledgments

The paper includes the solution results of the Ministry of Education, Youth and Sport research project No. MSM 0021630528.

7 References

[1] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II (Stiff and Differential-Algebraic Problems)*. Springer-Verlag Berlin Heidelberg, 1996. ISBN 3-540-60452-9.

[2] Web sites, High performance computing. `http://www.itsolutions.cz/TKSL`.

[3] J. Kunovský. *Modern Taylor Series Method*. FEI-VUT Brno, 1994. Habilitation work.

[4] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I*. vol. Nonstiff Problems. Springer-Verlag Berlin Heidelberg, 1987. ISBN 3-540-56670-8.