PROJECTILE MOTION – BOUNDARY VALUE PROBLEM AND OPTIMIZATION IN EDUCATION

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Abstract

The numeric treatment of boundary- and optimization problems is carried out by iterations. With an air resistance proportional to the square of the velocity, the projectile motion does not have any elementary analytic solution. In case of no air resistance however those solutions exist and they may be taken as tests for the numeric methods. The projectile motion with air resistance is characterized by a more steeply falling trajectory. Depending on the starting angle at constant initial velocity, the projectile motion as a boundary value problem and the maximization of the trajectory range are treated. The target deviation forms the functional of the boundary value problem. The algorithms consist of a modified Newton's Method, which is used to search the zero of the deviation, as well as of an angle correction proportional to the deviation. Optimization methods are the Three-Points Plan and the Method of the Golden Ratio. This method needs only a single new run for the comparison. During the maximization the uncertainty interval of the starting angle decreases iteratively by comparisons of the trajectory ranges. When the object hits the ground, i.e. the trajectory crosses the threshold zero, the simulation run is finished by a state event. The calculated range is then used as input parameter of the iteration. The algorithms are implemented in the TERMINAL section of the simulation system ACSL.

Keywords: Projectile motion, Education, Quadratic friction, Modified Newton's Method, Golden Ratio Method, Three-Points Plan.

Presenting author's biography

Rüdiger Hohmann completed his physics study at the Technical University of Dresden, afterwards he was employed in the Research and Development Department of 'veb rechenelektronik glashütte', a producer of electronic analog computers. Later Rüdiger Hohmann became assistant professor in the Institute of Mathematics at the Technical University of Magdeburg and after achieving his PhD he became group leader for hybrid simulation in the Computer Technique and Data Processing Department. He received his Habilitation from the Faculty of Computer Science, and is now among others lecturer for continuous simulation.

1 Introduction

The projectile motion is a suitable physical problem to give an introduction in iterative numeric methods for boundary value and optimization problems in courses. It is intuitively well comprehensible. The considered projectile motion to a target is a boundary value problem. The maximization of the trajectory range is an optimization problem. It is assumed that the air resistance will increase squarely with the flight velocity. Due to this nonlinearity there are no simple analytic solutions. Without air resistance the equation of motion problems are solvable in closed form and can be used to test the numeric methods. The initial trajectory velocity will be assumed as constant and so the parameter is the start angle.

This paper will introduce two methods to solve the boundary value problem and furthermore two methods to solve the optimization problem. A simulation ends with the zero crossing of the altitude y, the achieved trajectory ranch will than be used as input for the algorithm.

2 Equations of Motion

The equations of motion can be derived from Fig. 2. The motions in horizontal and vertical direction are considered as independent. The two equations (1) are connected by the air resistance rv^2.

$$
\begin{aligned}
m\frac{d^2x}{dt^2} &= -rv\frac{dx}{dt}, \quad x(0)=0,\ \dot{x}(0)=v_0\cos(\varphi_0) \\
m\frac{d^2y}{dt^2} &= -rv\frac{dy}{dt} - mg, \quad y(0)=0,\ \dot{y}(0)=v_0\sin(\varphi_0)
\end{aligned}
\tag{1}
$$

with

$$
\cos(\varphi)=\frac{\dot{x}}{v}, \quad \sin(\varphi)=\frac{\dot{y}}{v} \quad \text{und} \quad v=\sqrt{\dot{x}^2+\dot{y}^2}
$$

Parameter:

$v_0 = 100$ initial velocity

$g = 9{,}81$ gravitional acceleration

$m = 10$ mass

$r = 0{,}003$ friction coefficient

$x_1 = 800$ target

3 Boundary Value Task

The starting angle which leads the object to the target point, i.e. the boundary value x_1, is requested. This angle can be determined by a modified Newton's Method as well as by corrections which are proportional to the deviation from the target. There are two solutions φ_{01} and φ_{02}. The five steps of the Newton's Method for 10^{-4}-accuracy are shown in Tab.1 and Fig.1 shows the trajectory pairs.

3.1 Modified Newton's Method

The deviation in trajectory range $u = x - x_1$ from the target x_1 forms the functional of the method. It depends on the starting angle φ, which means $u = f(\varphi)$. The relevant angle $\varphi_0 = \varphi(0)$ has to be determined, so that $f(\varphi_0) = 0$. We have therefore a zero! Because $f(\varphi)$ is not an analytic function, the derivation $u' = df/d\varphi$ must be replaced in Newton's Method [1] by a difference approximation. This is generated by an additional run before the (k+1). iteration step. A little changed angle $\varphi_k(0)+\Delta$ leads to $\tilde{u}_k = f(\varphi_k(0)+\Delta)$. Using $u_k = f(\varphi_k(0))$, follows

$$
u'_k \approx \frac{\tilde{u}_k - u_k}{\Delta} \tag{3.1}
$$

as approximation of the derivative [2]. The best approximation can be achieved with a very small Δ. Newton's Method has the form now

$$
\varphi_{k+1}(0) = \varphi_k(0) - \frac{\Delta}{\tilde{u}_k - u_k}u_k \tag{3.2}
$$

The solving process alternates between the determination of u_k and $\tilde{u}_k$ with the Newton step which then delivers $\varphi_{k+1}(0)$. The change is organized by means of an integer variable. It is terminated with reaching the margin of error $|u_k| \le \varepsilon$.

Tab. 1 Newton's iteration steps

	t	phi0 [deg]	x	y
0	13.599	45.000	829.299	0.000
1	16.532	59.961	709.392	0.000
2	15.450	53.827	781.833	0.000
3	15.086	51.960	797.589	0.000
4	15.027	51.663	799.814	0.000
5	15.022	51.639	799.988	0.000

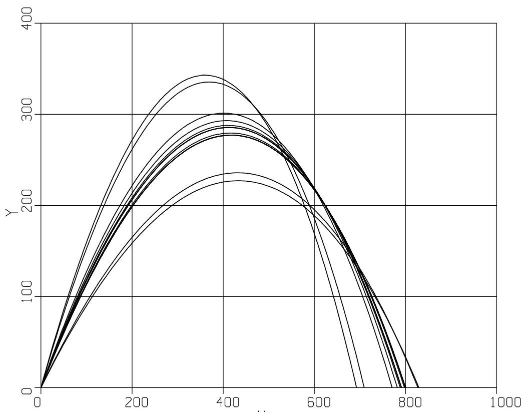

Fig. 1 Newton's target iteration

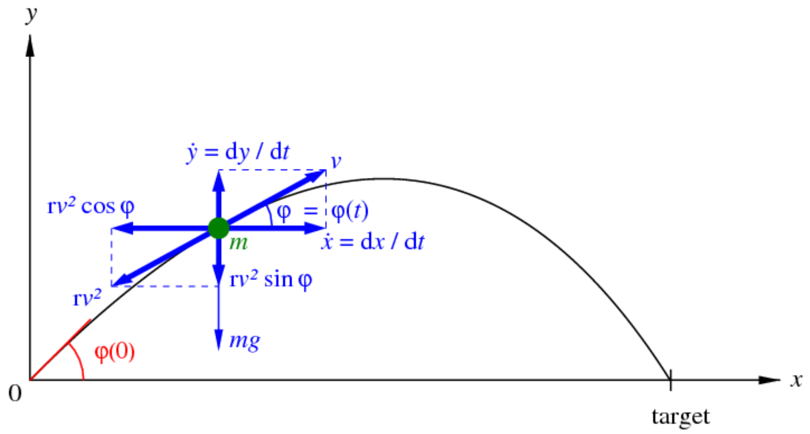

Fig. 2 Projectile motion (target throw)

3.2 Proportional Correction

A very simple method is to correct the angle proportional to the deviation of the target. But this method is not as robust as Newton's Method.

$$\varphi_{k+1} = \varphi_k + c\,(x_k - x_1) \qquad (3.3)$$

Whether the method converges or not, heavily depends on the size of the correction coefficient *c*. In the example it converges with $c = 0{,}001$.

3.3 Analytical Solutions

Without air resistance we are able to determine analytically both angles which hit the mark:

$$\varphi_{01} = \frac{1}{2}\arcsin\frac{g \cdot x_1}{v_0^2} \in (0, \pi/2), \qquad \varphi_{02} = \frac{\pi}{2} - \varphi_{01} \quad (3.4)$$

With the values above, we obtain $\varphi_{01} = 25{,}85$ degrees and $\varphi_{02} = 64{,}15$ degrees. The numerical Newton's iteration delivers these results beginning with 30 degrees and end at 60 degrees after five steps. With air resistance, the two angles are moving closer together: $\varphi_{01} = 35{,}27$ degrees, $\varphi_{02} = 51{,}64$ degrees.

For the free fall case with quadratic air resistance and initial values $y(0) = y_0$, $\dot{y}(0) = 0$, there is a closed-form solution [3]:

$$y(t) = y_0 - \frac{m}{r}\ln\cosh\left(\sqrt{rg/m} \cdot t\right) \qquad (3.5)$$

4 Optimization

After the target throw, the objective of maximizing the trajectory range is now the subject of consideration. As optimization methods we are using the Method of Golden Ratio that requires only one new run each time and the Three-Points Plan. During the maximization the uncertainty interval (tolerance) of the angle is reduced stepwise by comparing the trajectory ranges. It is assumed there is a unimodal function f. With a maximum at m in the interval between a and b, the function f is then monotonically increasing between a and m and monotonically decreasing between m and *b*.

4.1 Method of Golden Ratio

For unimodal functions exists a process that determines the maximum with $O(\log(L/T))$ evaluations of the function. T is the tolerance and L the length of the interval. The idea is, to evaluate f at two intermediate points x and y in the interval (with $x < y$). If then $f(x) > f(y)$ holds, the maximum must lie between a and y. But if $f(x) < f(y)$ holds, it is located in the interval between x and b. If x and y are in the middle range of the interval, we are able to halve the uncertainty interval with two evaluations, as shown in Fig. 3. By reducing iteratively, we get an interval smaller than T with $O(\log(L/T))$ steps [4].

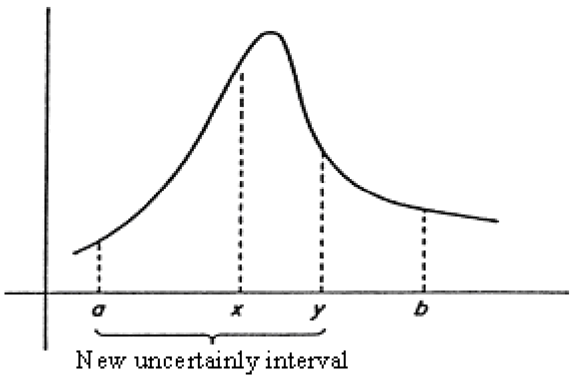

Fig. 3 Method of Golden Ratio

The Method of the Golden Ratio allows us to select the intermediate points x and y so that we are able to use one of the two values from the previous step again. The intermediate points divide the interval in a certain fixed ratio. x divides the range between a and b with the ratio *p*, y does the same with the ratio q:

$$\begin{aligned} x - a &= p(b - a) \\ y - a &= q(b - a) \end{aligned} \qquad (4.1)$$

The value q is the reciprocal of the golden ratio, which gave the method its name.

$$q = (\sqrt{5} - 1) / 2 \approx 0{,}618 \tag{4.2}$$

The value p is set as q^2. If $f(x) > f(y)$ applies, the new uncertainty interval lies between a and y, x serves as a "y-point" for this interval, and we have to calculate a new "x-point". If $f(x) < f(y)$ applies, the new uncertainty interval is located between x and b. y serves as an "x-point" for this interval and we have to calculate a new "y-point".

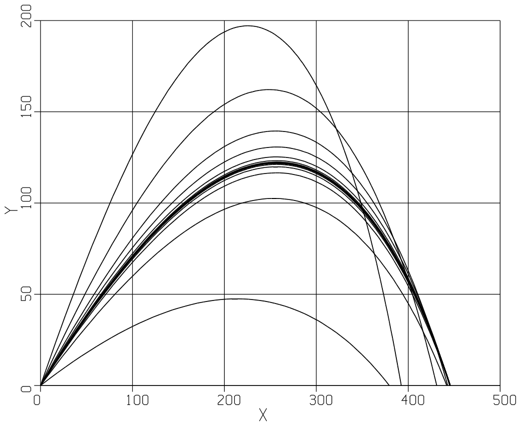

Fig. 4 Maximizing by Golden Ratio

The iterative process reduces the interval until the endpoints are closely enough to each other. In this uncertainty interval lies the desired value.

The angles $a = 0$ degree and $b = 90$ degrees are the boundaries of the search interval here. In this case the values at the angles $x = 34{,}38$ degrees and $y = 55{,}62$ degrees are compared in the first iteration step. The new search interval has a breadth of 55,62 degrees.

How many reductions steps are needed to achieve a tolerance T of the original uncertainty interval $L = (b - a)$ with the Golden Ratio Method? For this consideration it's relevant that both possible following intervals have the same size, reduced by q.

$$(y - a) = (b - x) \tag{4.3}$$

After n steps the uncertainty interval is reduced on the length $L_n = q^n L$. It is required that $L_n \le T$ or

$$q^n L \le T \tag{4.4}$$

Using the quadratic definition equation [4] for the golden ratio q

$$\frac{1}{q} = q + 1 \approx 1{,}618 \tag{4.5}$$

the logarithmic term (4.6) results from (4.4)

$$n \ge \frac{\log(L/T)}{\log(q+1)} \tag{4.6}$$

The integer variable n is of the order $O(\log(L/T))$. In the following we will use the common decadic logarithm $\lg(x)$ for numeric analysis.

The number m of evaluations (runs) at this method is $m = n + 2$, with two runs at the beginning. In the implementation both values $f(x)$ and $f(y)$, which are to be compared, are written in the output file before testing the termination condition. Then the test is carried out with the current uncertainty interval. After this we decide about the new reduced uncertainty interval. The next greater integer value of (4.6) is given by the ceiling function $\lceil n \rceil$:

$$m = \left\lceil \frac{\lg(L/T)}{\lg(q+1)} \right\rceil + 2 \tag{4.7}$$

$T = 10^{-3} L$ and $q = 0{,}618$ leads to $m = 17$ runs. In Tab. 2 the $n = 15$ reduction steps are represented for an air resistance of $r = 0{,}02$. To achieve a tolerance of $T = 10^{-4} L$, $m = 22$ runs would be necessary. These data were confirmed by simulations. Fig. 4 shows the accompanying function curves.

The sequence of the runs is organized by Block-IF-statements and an integer variable which corresponds to the indices of the function values $f_1 = f(x)$ and $f_2 = f(y)$. This part of the simulation takes place in the TERMINAL section of the simulation system ACSL [6] along with the output statements and the termination condition.

Tab. 2 Steps with Golden Ratio

	f1	f2	a [deg]	b [deg]
0	441.969	392.412	0.000	90.000
1	379.431	441.969	0.000	55.623
2	441.969	442.969	21.246	55.623
3	442.969	430.984	34.377	55.623
4	445.650	442.969	34.377	47.508
5	445.438	445.650	34.377	42.492
6	445.650	445.062	37.477	42.492
7	445.740	445.650	37.477	40.576
8	445.690	445.740	37.477	39.392
9	445.740	445.730	38.208	39.392
10	445.731	445.740	38.208	38.940
11	445.740	445.740	38.488	38.940
12	445.738	445.740	38.488	38.767
13	445.740	445.741	38.595	38.767
14	445.741	445.741	38.661	38.767
15	445.740	445.741	38.661	38.727

4.2 Three-Points Plan

This method compares three function values f at equidistant points. Initially these are the value in the interval center $f_1 = (a+b)/2$ and the two values at the boundaries $f_2 = f(a)$ and $f_3 = f(b)$. It follows the comparison of the maximum value with two new function values at half distances $\Delta x = \pm(a+b)/4$.

Function values outside the interval are not considered anymore. To obtain an uncertainty interval $2\Delta x$ of $(b-a)/2^n$, we require $(2n+3)$ values, which are here e.g. 17 points for $(b-a)/128$. The Three-Points Plan was also implemented on a hybrid computer [5].

It shall be proven that the number of evaluations is from the order $O(\log(L/T))$ as well. The uncertainty interval bisects itself with every step. This decrease is more rapid than with the Golden Ratio Method using a factor $\approx 0{,}618$, so that

$$(1/2)^n L \le T \tag{4.8}$$

The distance of the borders $L = (b-a)$ forms the initial uncertainty interval. To achieve the tolerance T are now

$$n \ge \frac{\log(L/T)}{\log 2} \tag{4.9}$$

reduction steps necessary. Every step requires two runs, except for the beginning step which requires three runs in total, one for each border and one for the middle of the interval. This leads to $m = 2n+3$ runs. Again the decadic logarithm is used:

$$m = 2\left\lceil \frac{\lg(L/T)}{\lg 2} \right\rceil + 3 \tag{4.10}$$

The tolerance of $T = 10^{-3} L$ requires $m = 23$ runs, and for $T = 10^{-4} L$ are here $m = 31$ evaluations necessary. This is easy to verify, because 10 steps reduces L to $L \cdot 2^{-10} = L/1024 < 10^{-3} L$ and 14 steps leads to the interval $L \cdot 2^{-14} = L/16384 < 10^{-4} L$.

If the maximum is in one of the borders of the search interval, the number of runs is nearly halved. Only the function values within the interval must be determined. In this special case is

$$m = \left\lceil \frac{\lg(L/T)}{\lg 2} \right\rceil + 3 \tag{4.11}$$

With the tolerances above this would be only $m = 13$ (Tab. 3) and $m = 17$ runs. Is the maximum only close to the border the number of runs rises again. It defines an area

$$\left\lceil \frac{\lg(L/T)}{\lg 2} \right\rceil + 3 \le m \le 2\left\lceil \frac{\lg(L/T)}{\lg 2} \right\rceil + 3 \tag{4.12}$$

Arguments outside the uncertainty interval should be implemented with negative function value to make them distinguishable. These cases exist only for the Three-Points Plan. The Method of the Golden Ratio compares only data within the interval.

To test the Three-Points Plan for a maximum on the boarder, a projectile motion without air resistance $r = 0$ is chosen. The optimal start angle is here 45 degrees, so that search intervals e.g. 20 degrees up to 45 degrees or 45 degrees up to 70 degrees get the optimal angle at the right or left border. In Tab. 3 the maximum is on the left border. The pseudo-function values of –1.000 for f_2 indicate that the arguments are outside of the definition area.

Tab. 3 Steps with Three-Points Plan

	f2	f1	f3	a [deg]	b [deg]
0	1019.368	923.861	655.237	45.00	70.00
1	-1.000	1019.368	995.205	38.750	51.250
2	-1.000	1019.368	1013.309	41.875	48.125
3	-1.000	1019.368	1017.852	43.438	46.562
4	-1.000	1019.368	1018.989	44.219	45.781
5	-1.000	1019.368	1019.273	44.609	45.391
6	-1.000	1019.368	1019.344	44.805	45.195
7	-1.000	1019.368	1019.362	44.902	45.098
8	-1.000	1019.368	1019.366	44.951	45.049
9	-1.000	1019.368	1019.368	44.976	45.024
10	-1.000	1019.368	1019.368	44.988	45.012

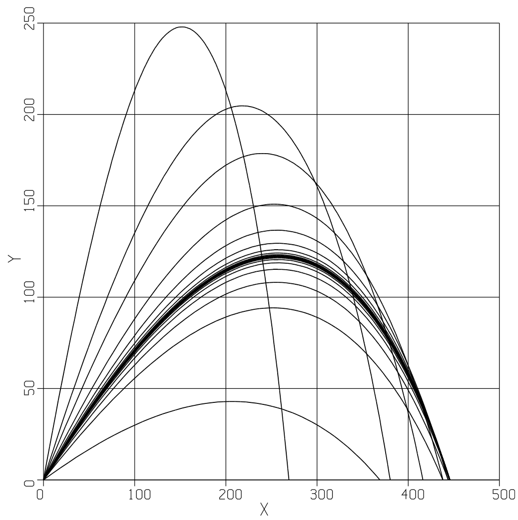

Fig. 5 Maximizing by Three-Points Plan

For a tolerance $T = 10^{-4} L$ and a high air resistance of $r = 0{,}02$ the trajectories are given in Fig. 5. The smallest and largest altitude of the curves are for the border values $\varphi_0 = 20$ degrees and $\varphi_0 = 70$ degrees.

4.3 Analytical Solution

The trajectory range without air resistance $r = 0$ is a known closed-form expression

$$x_1(\varphi) = \frac{v_0^2}{g} \cdot \sin(2\varphi) \; . \qquad (4.13)$$

It reaches its maximum at $\varphi = \pi / 4$, where $\sin(\pi / 2) = 1$. With the above parameters, the trajectory range is 1019.37. For $r = 0{,}02$ it is reduced on 455,74, the angle φ_0 on 38,71 degrees (see Tab. 2, Fig. 4 and Fig. 5). The projectile motion with air resistance is characterized by a more steeply falling trajectory.

5 Conclusions

Lectures on “Continuous Simulation” start with parameter studies by means of separate runs, iterative methods on the other hand require a coordinated series of runs. The numeric treatment of boundary- and optimization problems is carried out by means of such iterations.

The projectile motion is an easily understandable physical problem that does not have any elementary analytical solution however due to the quadratic air resistance. That is why simulations are useful. The boundary value and optimization problems of projectile motion and trajectory range offer themselves as exemplary cases. The iterative numerical algorithms can be elegantly checked by the frictionless cases, which have closed-form solutions. We are considering two algorithms for both tasks.

The algorithms are implemented in the TERMINAL section of the ACSL simulation system. From this section the program switches back to the start, the INITIAL section, until the current run achieves the termination condition.

The order of the solution steps is given by an integer variable. For Newton’s Method the variable can be of two values which stand for the determination of the deviation to the target and the approximation of the derivative. There are no different cases for the proportional correction of the boundary value, therefore it is a suitable exercise.

For the optimization with the Three-Points Plan the variable can be of three values, which are corresponding to the function values to compare. Since the arguments can be beyond the interval in this case, more case distinctions are necessary. In contrast the Golden Ratio Method compares only two function values in the middle of the search interval, so that a consideration of the boundaries is not required. Particularly a former function value can be reused. The choice of the arguments of the Golden Ratio Method is also mathematically of interest. Both optimization methods reduce a given search interval, in contrast to the known Gradient Methods, which steps forward in direction of the gradient.

The program structure can be transferred to other simulation systems if they have in addition to the section for numeric integration (DERIVATIVE section in ACSL), a section in which the logic operations of the methods can be carried out after a run. Parameters are given back to the start with changed values by jump statements.

The Java based simulation system AnyLogic permits e.g. automatic parameter studies and also the optimization. Out of a MATLAB program a simulation with SIMULINK can be called. This is the most flexible way to carry out a simulation for different parameter values. The execution of the program can be controlled by IF-statements. In FORTRAN 77 of ACSL the logic IF-statement, the Block-IF-statement and the jump statement GOTO are in use.

For the education a good intuitive understanding of the model-based physics is very important. Then all the attention can be focused on the simulations and their algorithms.

6 References

[1] I. N. Bronstein et al. Teubner-Taschenbuch der Mathematik. B. G. Teubner Verlagsgesellschaft Leipzig, 1996.

[2] J. Heinhold, and U. Kulisch. Analogrechnen: Eine Einführung. B · I · Hochschultaschenbücher 168/168a*. Bibliographisches Institut Mannheim, 1969.

[3] M. Zöllner. Computeralgebrasysteme in der Simulation: Vergleichende Untersuchung von symbolischen und numerischen Lösungen mit Mathematica. Diplomarbeit im Studiengang Informatik, Magdeburg, 2006.

[4] H. Abelson und G. J. Sussman mit J. Sussman. Struktur und Interpretation von Computer-programmen: Eine Informatik-Einführung. Springer-Verlag, 1991.

[5] R. Hohmann, and L. Nowak. Hybride Parameter-optimierung nach einer diskreten, optimalen Gradientenmethode. Wiss. Z. Techn. Hochsch. Magdeburg, 24 (1980) Heft 3.

[6] ACSL Model. ACSL Reference Manual, Version 11. MGA Software, 1995.